Pigeonhole Principle

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The *pigeonhole principle* states that if $n+1$ pigeons are placed into n pigeonholes, at least one of the pigeonholes will contain at least two pigeons. While this result might seem obvious, this principle can be applied in clever ways to solve olympiad-style problems. Usually, the difficulty in solving pigeonhole problems comes from choosing the pigeons and the pigeonholes.

Problem 1.

(a) Every human has at most 500 thousand hairs on their head. Explain why this means that there are two people in Bhutan with the same number of hairs on their heads.

(b) Prove that there are one thousand people in Bhutan that all share a birthday.

(c) Holden has 7 pairs of white socks, 3 pairs of black socks, 2 pairs of gray socks, and 1 pair of turtle socks in his drawer. How many socks does Holden need to take out of his drawer to guarantee a matching pair?

Problem 2.

(a) 31 students in a mathematics camp take a 3 question exam, where each question is graded out of 7 points. Prove that at least 2 students will receive the same score.

(b) 31 students in a mathematics camp take an exam. In total, 400 questions were solved. Prove that two students solved the same number of questions.

(c) 31 students in a mathematics camp take a 3 question exam. Prove that at least 4 students will solve the same set of problems.

Problem 3.

(a) There are 5 points in an equilateral triangle with side length 2. Prove that two of the points are at most 1 unit apart.

(b) There are 10 points in a square with side length 3. Prove that two of the points are at most $\sqrt{2}$ units apart.

Problem 4. Prove that for every positive integer n, there is a multiple of n whose digits consist of only zeroes and ones.

Problem 5.

(a) Prove that among any 109 integers, two of them differ by a multiple of 108.

(b) Prove that if 109 integers are placed in a line, it is always possible to select a contiguous block whose sum is a multiple of 108.

(c) Prove that if 108 integers are placed in a line, it is always possible to select a contiguous block whose sum is a multiple of 108.

Problem 6. At a party with 10 people, several pairs of people shook hands with each other. Prove that two people shook hands with the same number of people.

Problem 7.

(a) Prove that among any 11 numbers in $1, 2, \ldots, 20$, there are two numbers with a difference of 10.

(b) Prove that among any 11 numbers in $1, 2, \ldots, 20$, two of the numbers have a greatest common divisor of 1.

(c) Prove that among any 11 numbers in $1, 2, \ldots, 20$, one number is a multiple of another number.

Problem 8. In a 8×8 grid, 25 rooks are put on distinct cells. Prove that one can select four rooks such that no two are in the same row or column.