§ Problem Statement

2025 black and white stones are arranged in a 45×45 grid. A set of four stones is *special* if the four stones form a 2×2 square, the top-left stone and the bottom-right stone are both black, and the bottom-left stone and top-right stone are both white. A *move* consists of inverting the colors of any special set of four stones.

(a) Prove that it is impossible to make infinitely many moves.

(b) Across all possible initial configurations, what is the maximum number of moves that can be made?

§ Solution

The maximum possible number of moves is $(22 \cdot 23)^2 = \boxed{256036}$.

Upper bound

To prove that at most $(22 \cdot 23)^2$ moves can be made, label the 45 columns from left to right with the integers -22 to 22, and label the 45 rows from bottom to top with the integers -22 to 22. Define the *weight* of a position to be the product of its row label and its column label, and define the *energy* of a configuration to be the sum of the weights of the black stones in the configuration.

Claim 1. *Making a move always increases the energy by exactly one.*

Proof. Applying a move on a special set of four stones with row labels $\{a, a+1\}$ and column labels $\{b, b+1\}$ increases the sum of the weights of the black stones by

$$ab + (a+1)(b+1) - a(b+1) - b(a+1) = 1. \qquad \square$$

The minimum possible initial energy is

$$\sum_{a=-22}^{-1} \sum_{b=1}^{22} ab + \sum_{a=1}^{22} \sum_{b=-22}^{-1} ab = -2(1 + 2 + \ldots + 22)^2 = -\tfrac{1}{2}(22 \cdot 23)^2,$$

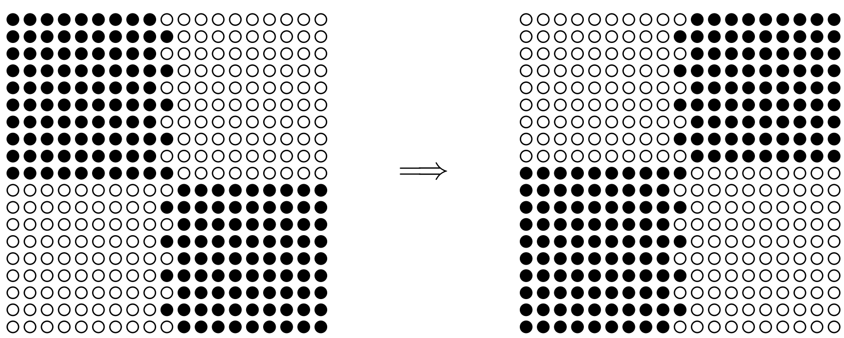
and the maximum possible final energy is

$$\sum_{a=-22}^{-1} \sum_{b=-22}^{-1} ab + \sum_{a=1}^{22} \sum_{b=1}^{22} ab = 2(1 + 2 + \ldots + 22)^2 = \tfrac{1}{2}(22 \cdot 23)^2.$$

Therefore, Claim 1 implies that at most $(22 \cdot 23)^2$ moves can be made.

Lower bound

To prove attainability, consider the following initial and final configurations, which have energies of $-\frac{1}{2}(22 \cdot 23)^2$ and $\frac{1}{2}(22 \cdot 23)^2$ respectively. It suffices to show that the left configuration can be turned into the right configuration, by Claim 1.



Claim 2. *If two rows are adjacent and are color complements (i.e. every black stone in the top row corresponds to a white stone in the bottom row, and vice versa), then a sequence of moves can be made to result in a configuration where the top row has its black stones on the right and the bottom row has its black stones on the left.*

Proof. Repeatedly make moves until no more can be made, and note that the rows remain color complements throughout the entire process. By construction, no white stone in the top row can lie to the left of a black stone in the top row; this forces the desired final configuration. □

Call a sequence of moves achieving the goal in Claim 2 a *row flip*. Since the number of black stones contained in any two adjacent rows is always 45, repeatedly applying row flips until no more can be made will result in a configuration where every black stone in the rightmost column is above every white stone in the rightmost column. Since row flipping forces each row to be color-segregated (i.e. a block of white stones followed by a block of black stones, or vice versa), the final configuration must be the one given in the figure above, as desired.

Upper bound (alternate proof)

Label the rows and columns using the integers $1, \ldots, 45$.

Claim 3. *Let $\mathcal{M}(a,b)$ denote the number of times that the 2×2 square spanning row labels $\{a, a+1\}$ and column labels $\{b, b+1\}$ can be moved, and define $\mathcal{M}(a,b) = 0$ if $a = 0$ or $b = 0$. Then*

$$\mathcal{M}(a,b) + \mathcal{M}(a-1,b-1) - \mathcal{M}(a-1,b) - \mathcal{M}(a,b-1) \le 1$$

for every pair of positive integers (a,b).

Proof. The number of times that the stone in with coordinates (a,b) is flipped from white to black is $\mathcal{M}(a-1,b-1) + \mathcal{M}(a,b)$, and the number of times that the stone is flipped from black to white is $\mathcal{M}(a-1,b) + \mathcal{M}(a,b-1)$. The bound follows after observing that the difference of these quantities must be at most 1. □

Now,

$$\begin{aligned}
&\sum_{a=1}^{22}\sum_{b=1}^{22}\mathcal{M}(a,b) \\
&= \sum_{a=1}^{22}\sum_{b=1}^{22}((23-a)(23-b) + (23-a)(22-b) - (23-a)(22-b) - (22-a)(23-b))\mathcal{M}(a,b) \\
&= \sum_{a=1}^{22}\sum_{b=1}^{22}(23-a)(23-b)\left(\mathcal{M}(a,b) + \mathcal{M}(a-1,b-1) - \mathcal{M}(a-1,b) - \mathcal{M}(a,b-1)\right) \\
&\le \sum_{a=1}^{22}\sum_{b=1}^{22}(23-a)(23-b) = (1 + \ldots + 22)^2 = \tfrac{1}{4}(22 \cdot 23)^2,
\end{aligned}$$

where the last inequality follows from Claim 3. Therefore,

$$\begin{aligned}
&\sum_{a=1}^{44}\sum_{b=1}^{44}\mathcal{M}(a,b)\\
&=\sum_{a=1}^{22}\sum_{b=1}^{22}\mathcal{M}(a,b)+\sum_{a=1}^{22}\sum_{b=23}^{44}\mathcal{M}(a,b)+\sum_{a=23}^{44}\sum_{b=1}^{22}\mathcal{M}(a,b)+\sum_{a=23}^{44}\sum_{b=23}^{44}\mathcal{M}(a,b)\\
&\le \tfrac{1}{4}(22\cdot 23)^2+\tfrac{1}{4}(22\cdot 23)^2+\tfrac{1}{4}(22\cdot 23)^2+\tfrac{1}{4}(22\cdot 23)^2\\
&=(22\cdot 23)^2
\end{aligned}$$

since the last inequality follows by symmetry from rotating the grid.

§ Variants

Variant A. n^2 black and white stones are arranged in a $n \times n$ grid, where n is a fixed positive integer. A set of four stones is *special* if the four stones form a 2×2 square, the top-left stone and the bottom-right stone are both black, and the bottom-left stone and top-right stone are both white. A *move* consists of inverting the colors of any special set of four stones. Across all possible initial configurations, what is the maximum number of moves that can be made?

Solution sketch. The answer is

$$\begin{cases} \frac{1}{16}n^4 & \text{when } n \text{ is even, and} \\ \frac{1}{16}n^2(n-1)^2 & \text{when } n \text{ is odd.} \end{cases}$$

When n is odd, the solution proceeds analogously to the one given above. When n is even, the proof of the lower bound requires labeling the columns and rows with the n fractions $\frac{1-n}{2}, \frac{3-n}{2}, \dots, \frac{n-3}{2}, \frac{n-1}{2}$ and defining the weight of a position to be the product of its row label and column label. The construction is simpler when n is even; the desired initial configuration consists of black stones in negative-weight positions and white stones in positive-weight positions.

Variant B. 2025 black and white stones are arranged in a 45×45 grid. A set of four stones is *special* if the four stones form the vertices of an axis-aligned rectangle, the top-left stone and the bottom-right stone are both black, and the bottom-left stone and top-right stone are both white. A *move* consists of inverting the colors of any special set of four stones. Across all possible initial configurations, what is the maximum number of moves that can be made?

Solution sketch. This is a stronger version of the original problem statement, but the proof idea is the same.

Variant C. ab black and white stones are arranged in an $a \times b$ grid, where a and b are fixed positive integers. A set of four stones is *special* if the four stones form a 2×2 square, the top-left stone and the bottom-right stone are both black, and the bottom-left stone and top-right stone are both white. A *move* consists of inverting the colors of any special set of four stones. Across all possible initial configurations, what is the maximum number of moves that can be made?

Solution. The answer is $\lfloor \frac{1}{4}a^2 \rfloor \lfloor \frac{1}{4}b^2 \rfloor$; the proof and construction are analogous to the one given in variant A. Care should be taken, depending on the parity of a and b.

Variant D. 45^3 black and white stones are arranged in a $45 \times 45 \times 45$ cube. A set of eight stones is *special* if

- the eight stones form the vertices of a $2 \times 2 \times 2$ cube,

- the front-top-left, front-bottom-right, back-bottom-left, and back-top-right stones are all black, and
- the remaining four stones are white.

A *move* consists of inverting the colors of any special set of four stones. Across all possible initial configurations, what is the maximum number of moves that can be made?

Solution. The answer is $(22 \cdot 23)^3$; a similar proof to the two-dimensional case works.

Remark. A similar idea can be used to solve the problem in any number of dimensions.

Variant E. n^2 black and white stones are arranged in a $n \times n$ grid, where n is a fixed positive integer. A set of four stones is *special* if the four stones form a 2×2 square, the top-left stone and the bottom-right stone are both black, and the bottom-left stone and top-right stone are both white. A *move* consists of inverting the colors of any special set of four stones. Across all possible initial configurations, let N be the maximum number of moves that can be made. For how many possible initial configurations is it possible to make N moves?

Solution. Since no positions have weight zero when n is even, the answer is 1 when n is even. I am not sure what the answer is when n is odd; The answer is 2 when $n = 1$, 8 when $n = 3$, and I suspect the answer is always 8 for all odd $n \geq 3$.

§ Comments

- The variants above can also be combined in various ways to create other interesting problems.
- Replacing 45 in the problem statement with an even number makes the problem easier because the construction for even n is easier to find.
- I came up with this problem during an Athemath office hours session. There were no students at the time, so I was bored and decided to write a grid combinatorics problem.

§ Metadata

This problem was selected as Problem 3 of the 2025 USEMO.

- Title: Stones on a Grid
- Author: Holden Mui
- Subject: combinatorics
- Description: black and white stones on a 45×45 grid; maximize the number of moves

- Keywords: color, grid, maximize, move, stones
- Difficulty: USAMO 2/5
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