§ Problem Statement

Let n be a positive integer. Prove that there is some positive integer k for which

$$a_1! + a_2! + \cdots + a_n! + k$$

is never square for any choice of positive integers $a_1, a_2, \ldots, a_n$.

§ Solutions

Solution A

The idea is to collect a system of congruences $k \equiv b_i \pmod{p_i}$ such that for every choice of $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, there is some index i for which $a_1! + \cdots + a_n! + b_i \pmod{p_i}$ is a quadratic nonresidue. Then, k can be constructed using the Chinese remainder theorem.

Claim 1. *For every nonnegative integer n, there is an integer k and modulus m for which*

$$a_1! + a_2! + \cdots + a_n! + k \pmod{m}$$

always a quadratic nonresidue for any choice of positive integers $a_1, a_2, \ldots, a_n$.

Proof. Induct on n; the base case $n = 0$ is true by taking $k = 2$ and $m = 3$. For the inductive step, suppose that there are integers k' and m' for which $a_1! + \cdots + a_{n-1}! + k'$ is always a quadratic nonresidue modulo m'.

If any $a_i \geq m'$, then $a_1! + \cdots + a_n! + k'$ will be a quadratic nonresidue modulo m' by the inductive hypothesis. Otherwise, all $a_i < m'$; for each such choice of $(a_1, \ldots, a_n)$, pick a new prime $p_{(a_1,\ldots,a_n)}$ and integer $k_{(a_1,\ldots,a_n)}$ for which $a_1! + \cdots + a_n! + k_{(a_1,\ldots,a_n)}$ is a quadratic nonresidue modulo $p_{(a_1,\ldots,a_n)}$. Choosing $k \pmod{m}$ to be the solution of the congruences $k \equiv k' \pmod{m'}$ and $k \equiv k_{(a_1,\ldots,a_n)} \pmod{p_{(a_1,\ldots,a_n)}}$ for all $(a_1, \ldots, a_n) \in \{1, 2, \ldots, m' - 1\}^n$ works, by the Chinese remainder theorem. □

Remark. Instead of choosing $k_{(a_1,\ldots,a_n)}$ and $p_{(a_1,\ldots,a_n)}$ for each choice of $(a_1, \ldots, a_n) \in \{1, 2, \ldots, m' - 1\}^n$, there is an alternative finish. Instead, choose a positive integer b such that $a_1! + \cdots + a_n! + b$ is never square for any choice of $a_1, \ldots, a_n < m$; such a b exists because the square numbers have arbitrarily long gaps. Then, choose a prime $p \nmid m'$ for which b is a quadratic nonresidue modulo p (such p exists by standard number theory facts). Then choosing $k \pmod{m}$ to be the solution of the congruences $k \equiv k' \pmod{m'}$ and $k \equiv b \pmod{p}$ works.

Solution B

It suffices to show that there is some positive integer m for which

$$a_1! + a_2! + \cdots + a_n! - s^2 \pmod{m!}$$

does not achieve every residue modulo $m!$. Indeed, there are m possible residues for each $a_i!$ and at most

$$m! \cdot \prod_{\text{odd prime } p \leq m} \frac{(p+1)/2}{p} \leq m! \left(\frac{2}{3}\right)^{\Theta(m/\ln m)}$$

possible residues for s^2 by the prime number theorem. Thus, there are

$$m^n \cdot m! \left(\frac{2}{3}\right)^{\Theta(m/\ln m)} = m! \cdot o(1)$$

possible residues modulo $m!$, as desired.

§ Metadata

This problem was selected as Problem 3 of the 2026 AMM.

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